Calculation of gravitational forces of a sphere and a plane

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Objective

The purpose of this paper is to evaluate two methods of gravity calculation in and around a sphere. Following a comparison and discussion of results, the gravitational forces of a plane are examined. All graphs and illustrations are based on an EXCEL sheet which has been programmed to use the discrete calculation method.

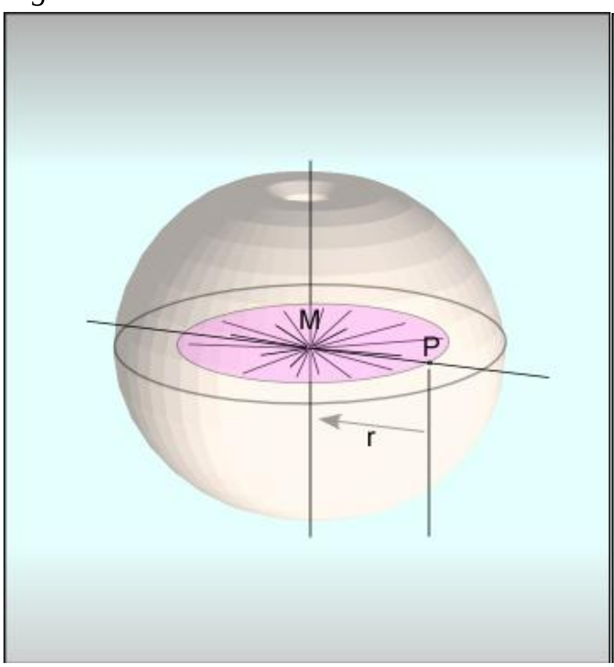
Fundamentals of the calculation

Two basic methods are available for the calculation of gravitational forces:

1. The most commonly used center oriented, integral method
2. Discrete, measuring point oriented method

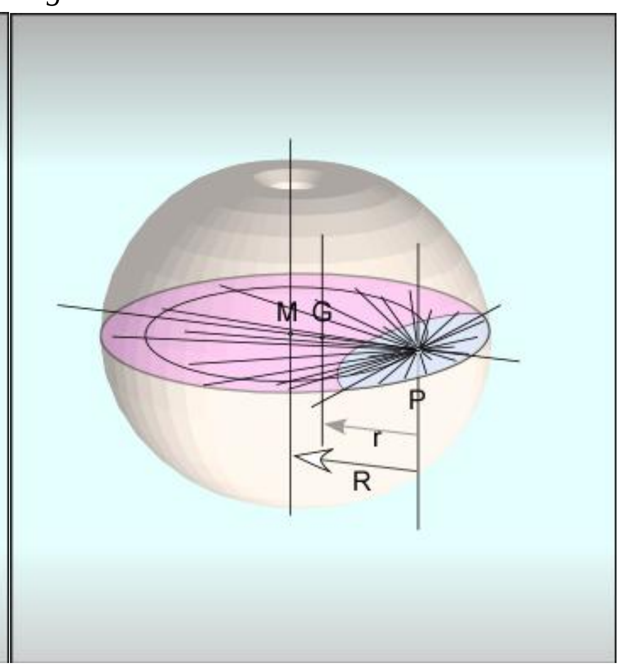
The integral, center oriented method relies solely on masses within the visual orbit around the center of a single mass point. The discrete calculation method, however, includes all masses within the sphere. The ECXEL sheet includes the single calculations for every mass point. These are then combined for further calculations.

Figure 1



Center-oriented integral

Figure 2



Measuring point-oriented discrete

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The calculation and comparison of results occur in a sphere with 5056 mass points (a plane with 357 mass points). Hence, starting from the center, each has 10 measuring points. Similar results are to be expected when the different methods are compared. A slight discrepancy caused by the nature of discrete calculation will occur, however, it should remain within single digit percent values.

Derivation of gravity with the center-oriented, integral method according to Alonso and Finn (2000)

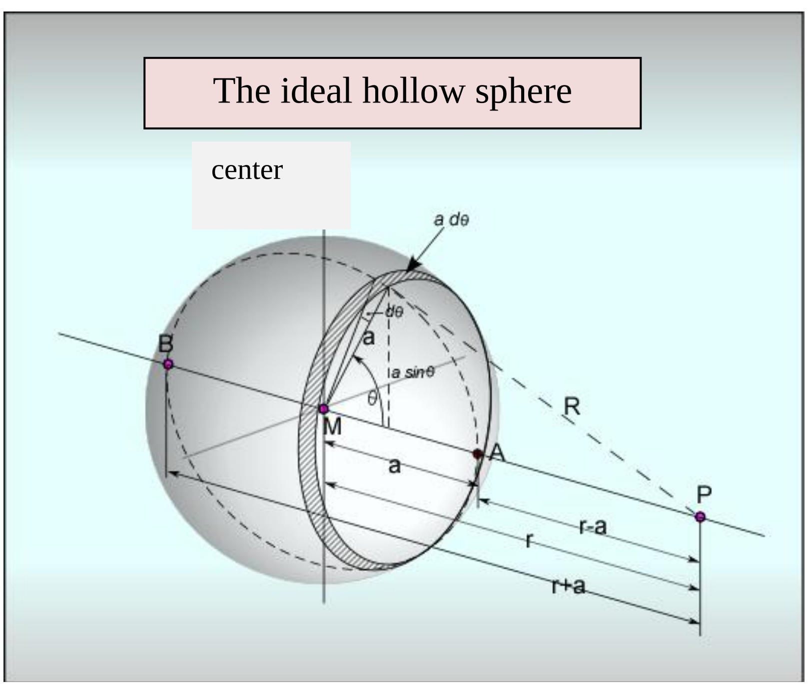
Students should be familiar with the calculation model, as it is portrait in every science school book. Alonso and Finn (2000) derive the formula for gravitational forces in the following way:

The formulas concerning gravitational forces pertain to single masses only. When determining the forces between planets and the sun, for example, the diameter of the planets is insignificantly small compared to the vast distances between them. Consequently, they can be used as single masses. Nevertheless, with a relatively short distance, the volumes and gravitational forces within the planets or spheres need to be included. Even Newton himself was unsure of how to accomplish this and postponed the publication of his works until he had found a solution - almost two decades later.

The gravitational forces between two masses are calculated with:

$$F = \frac{\gamma \times m \times M}{r^2} \qquad \textbf{(F1)}$$

Figure 3



Calculation of the gravitational field for a point outside of a mass that is evenly distributed over a hollow sphere

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The calculation of a hollow sphere's gravitational field usually occurs with the help of an outside mass P and the division of the sphere's surface into narrow strips. The center of these strips lies on the line between A and B, with the radius $R = a\sin\theta$ and the width $ad\theta$. Consequently, the surface of each strip is: length x width or

$$(2\pi a\sin\theta) \times (ad\theta) = 2\pi a^2 \times \sin\theta d\theta\,.$$

With *m* being the combined mass of the sphere, the mass per surface unit is $\frac{m}{4\pi a^2}$ and the mass of each strip is

$$\frac{m}{4\pi a^2} \times (2\pi a^2 \times \sin\theta \times d\theta) = \frac{1}{2} \times m \times \sin\theta \times d\theta.$$

The points of each strip are at an equal distance *R* from the point P. Hence, the gravitational potential of the strips on the point P is:

$$dV = \frac{\gamma(\frac{1}{2}m\times\sin\theta\times d\theta)}{R} = -\frac{\gamma\times m}{2R} \times \sin\theta \times d\theta \qquad \textbf{(F1.1)}$$

According to Figure 3, $R^2 = a^2 + r^2 - \cos\theta_0$ which means *r* and *a* are constant.

$$2R \times dR = 2ar \times \sin\theta \times d\theta \rightarrow \sin\theta \times d\theta = \frac{R\times dR}{a\times r}$$

which leads to

$$dV = -\frac{\gamma\times m}{2ar} dR \qquad \textbf{(F2)}$$

For the integral determination of the whole potential for the sphere surface, the borders for *R* are set as $r + a$ and $r - a$.

$$V = -\frac{\gamma\times m}{2ar}\int_{r-a}^{r+a} dR = -\frac{\gamma\times m}{2ar}(2r) = -\frac{\gamma\times m}{r} \text{ for all } r > a \qquad \textbf{(F3)}$$

If the point P is located within the sphere, the calculation looks as followed:

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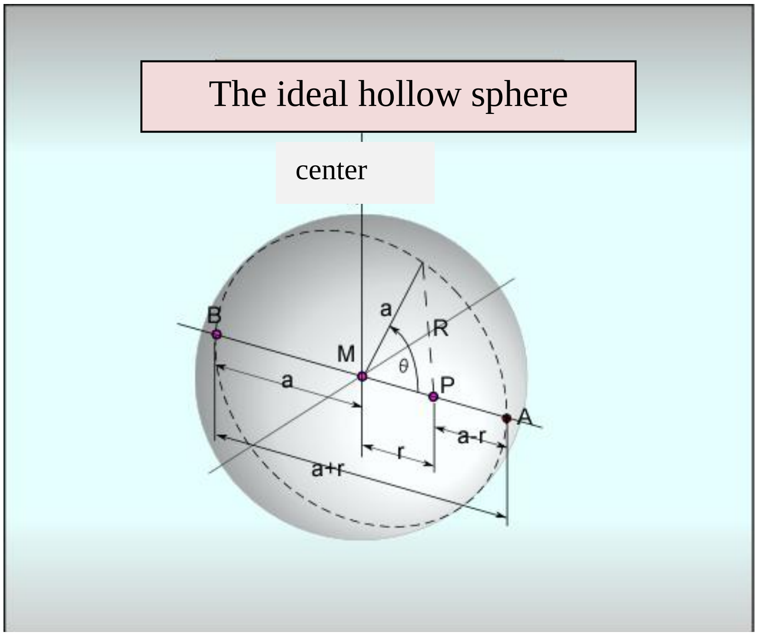
Figure 4



Calculation of the gravitational field for a point P inside of the ideal hollow sphere

The potential of the sphere with the radius *R* between $a + r$ and $a - r$ is

$$V = -\frac{\gamma \times m}{2ar} \int_{a-r}^{a+r} dR = -\frac{\gamma \times m}{2ar}(2r) = -\frac{\gamma \times m}{a} \text{ for } r < a \quad \textbf{(F4)}$$

This constant value is independent from the location of point P. The modification of (F1) leads to the formulas for the field force on the points outside of the sphere surface

$$G = -\frac{\gamma \times m}{r^2} e_r \qquad \text{for all } r > a \quad \textbf{(F5)}$$

and on the inside

$$G = 0 \qquad \text{for all } r < a \quad \textbf{(F6)}$$

A comparison of (F3) and (F5) shows

The field force and potential of masses are equal for a mass that is distributed evenly on the surface and a mass that is inside of a sphere. For all points P inside of a hollow sphere, the field force is 0, and the energy is constant.

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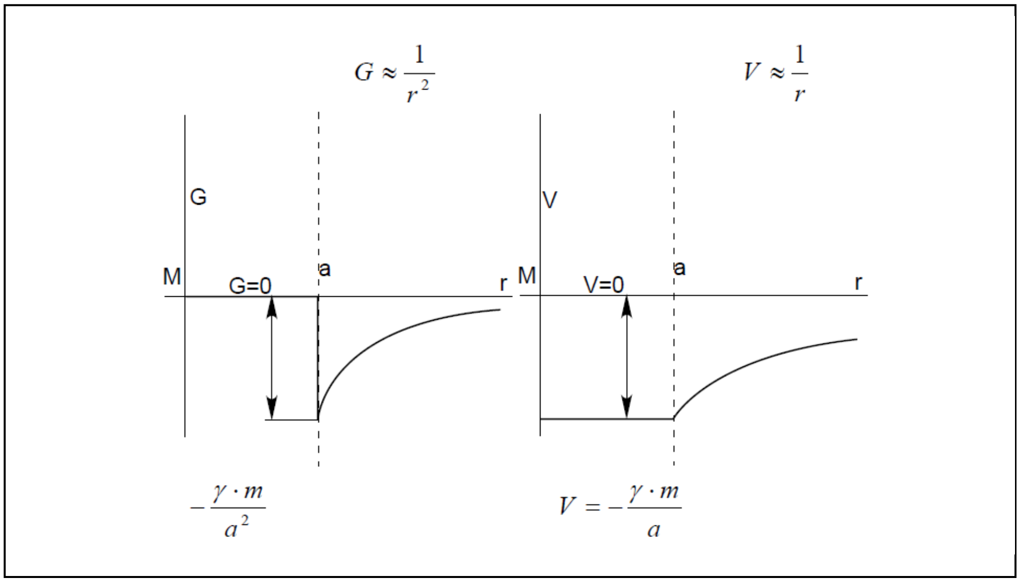
Figure 5

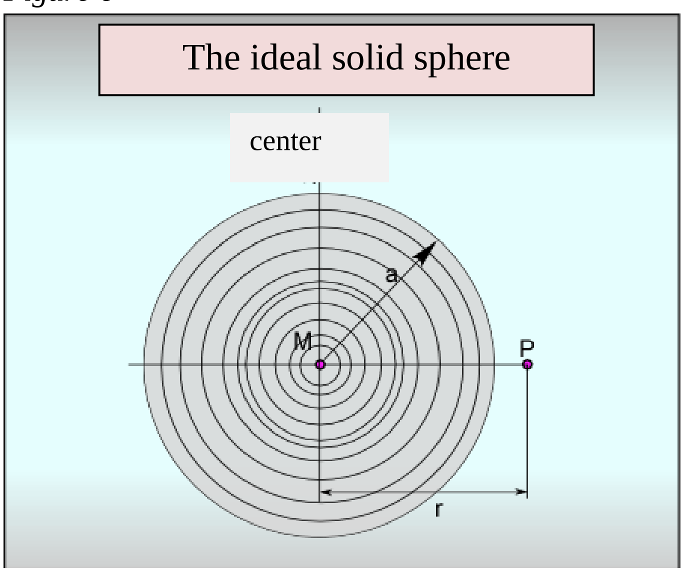


Differences of *G* and *V* as a function of distance from the center of a homogenously distributed mass on the surface of a sphere

Coming from the outside toward the center of the sphere, the potential remains constant when passing the boundary line of the sphere. (The slope, however, changes irregularly.) The gravity G changes immediately in accordance with formula (F5) and falls until it reaches a zero value in the center.

Gravity in and around a solid sphere

Figure 6



Calculation of a gravitational field for a point P outside of the solid sphere

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The mass of a solid sphere is $m = \frac{1}{4}\pi a^2\sigma$, if σ describes the mass density. Once the surface of the sphere is crossed, the force of the field is equal to $-4\pi\gamma\sigma$.

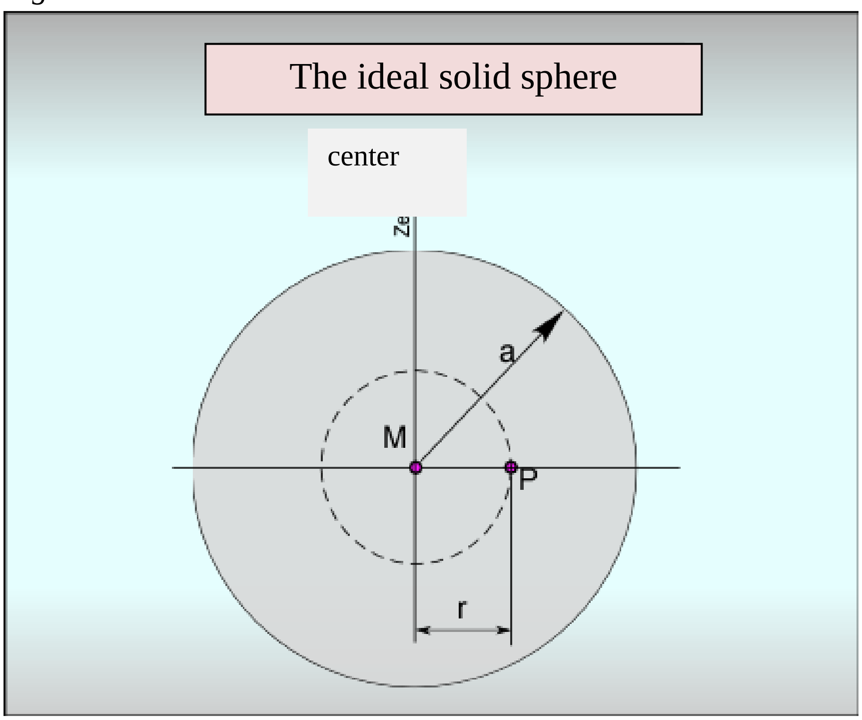
The same applies to the mass distribution in a solid plane. The calculation of gravitational forces of sphere and plane are similar.

Under the assumption that the sphere is completely solid and homogenous, we can treat it like a sum of thin layers. Each layer exerts a specific field force, which is determined with the formulas (F5) and (F6). Since, however, the slices have a common center, the distance *r* from P is the same in each calculation.

Therefore, the result of formula (F5) applies, and the entire mass of the sphere can be used as a single point mass in the center.

Furthermore, the sphere does not have to be homogenous, as long as the distribution of its masses is sphere symmetrical and independent from its direction.

Figure 7



Gravity calculation for a point P inside of a solid sphere

In order to determine the field force within a homogenous solid sphere, point P is located at a distance to the sphere's center that is smaller than its radius: $r < a$. The

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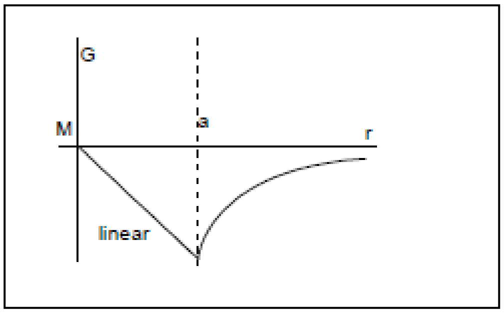
sphere slices beyond point P do not contribute to the field force on P. The slices between the center and P combined deliver a force equal to formula (F5). If *m*' is the mass of the sphere with the radius *r*, the field force in P is:

$$G = -\frac{\gamma \times m}{r^2} e \qquad \textbf{(F7)}$$

Figure 8



The graph in *Figure 8* illustrates the differences in G in a solid homogenous sphere as function from the distance to the sphere's center.
With the volume of a sphere defined as $\frac{4}{3} \times \pi \times a^3$, the mass *m*' can be calculated with

$$m' = \frac{m}{\frac{4}{3} \times \pi \times a^3} \left(\frac{4}{3} \times \pi \times r^3 \right) = \frac{m \times r^3}{a^3}.$$

The gravitational force or field force within a sphere is calculated with

$$G = -\frac{\gamma \times m \times r}{a^3} e^r$$

Consequently, the field force within a homogenous sphere depends directly on the distance from its center (see *Figure 8*).
At this point, Alonso and Finn (2000) leave it up to the reader to see that the gravity potential on a point outside of the sphere determined with (F4) weighs heavier into any calculation than that of a point on the inside.

$$V = -\frac{\gamma \times m}{2 a r^3} (r^2 - 3a^2) \qquad \text{for all } r < a$$

Even though errors occur if the examined body has a different symmetry, Alonso and Finn (2000) conclude that regardless of any problems with sphere symmetry, the main

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point is to understand that the gravitational characteristics of a homogenous sphere depend on the distance of P to the sphere's center. The solution of any other problems pertaining to gravity calculation is often a symmetry adjustment. Such a simplification usually turns the issue into an easy to solve math problem.

Alonso and Finn's conclusion

The purpose of these calculations and derivations is to see if the formulas for gravitational forces apply to point masses as well as masses with a larger volume.

All formulas pertaining to the calculation of gravitational forces are valid for point masses and masses with volume – under specific circumstances. Spheres can be used as point masses if only the masses within the orbit of P are considered. The masses beyond this orbit shall not be used. In addition, a sphere has to be rotation symmetrical.

Hence, *m* and *r* are known variables and can be used to determine *F*.

Derivation of gravity with the measuring point-oriented discrete calculation method

When calculating discretely, a few fundamental evaluations can provide new insight and understanding of gravitational forces. Alonso and Fin (2000) introduced two of their results, which have been summarized on the last four pages:

1. In the center of a point symmetrical, solid, homogenous sphere gravity is not noticeable.
2. As long as the masses on the surface of a hollow sphere are distributed evenly, the gravity on the inside is equal to zero.

What happens when the masses are **not** symmetrical?

→ 1. Gravity in the center of a point symmetrical sphere

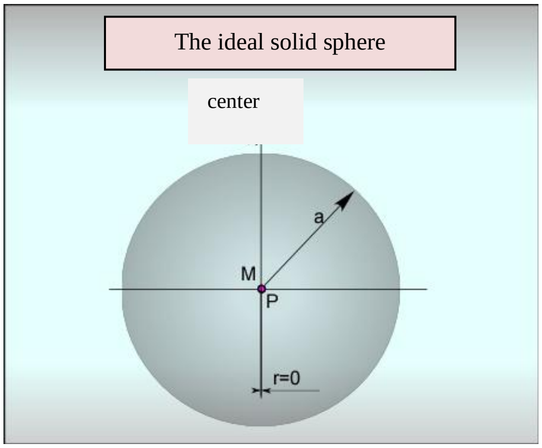

Figure 9
A point P is located in the center of a point symmetrical sphere shaped body. The mass distribution is even. The gravitational forces of the surrounding masses counteract each other, as long as they have the same distance from the center and are exactly opposite each other. Consequently, the gravity in the center is completely cancelled out.

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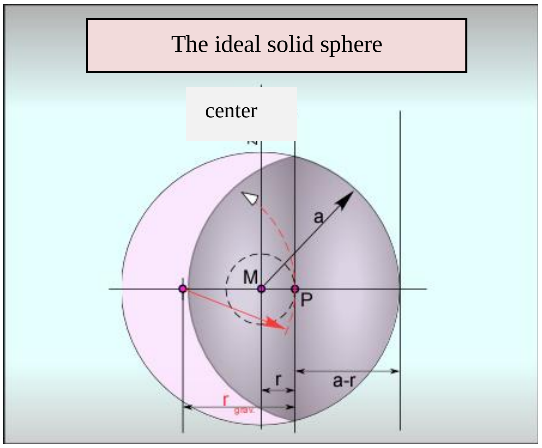

Figure 10
When moving point P toward the edge of the sphere, the cancellation of gravitational forces remains the same, as long as the masses have the same distance from point P and are exactly opposite each other. P is in the center of a lens shapes body that is free of gravity (only pertaining to P). This shape is symmetrical to the line that runs through P and is orthogonal to the radius of the sphere, as shown in a darker shade of gray. The height of the body is $a - r$. The subtraction of the lens shaped body from the sphere leaves a sickle shaped body that exerts gravitational forces onto point P. It is located to the left of the lens shaped body. Clearly, the masses influential to P are not symmetrical anymore; therefore, P is subjected to gravity. Surprisingly, the radius *r* and the mass *m* of the sphere are now irrelevant for further gravity calculations, as they are part of the gravity free lens. Of importance is only the sickle shaped body.
(Compare to the results of the integral calculation by Alonso and Finn, where *r* and *m* are substantial for the determination of gravity.)

→ 2. Gravity on the inside of a hollow sphere
According to the integral calculation method, the inside of a hollow sphere is free of gravity. *Figure 11* displays a hollow sphere centered within a solid sphere. The hollow (mass free) sphere is big enough to allow a full orbit of point P around its center. In addition, *Figure 11* shows the gravitational forces and radius. The radius *r* from the integral method has been turned into $r_{vis.}$, because the inner orbits of P are only visual orbits.
After evaluating the schematic illustration of all orbits and graphs, it should be clear that only the mass of the sickle shaped body outside of the hollow sphere exerts gravity on point P. The force of this gravity shall not be determined in this context. This is done and thoroughly explained in the “many-body problem in the calculation of galaxies” by Krause (2005a). Instead of using the masses, the forces of all measuring points are added and combined in a single rotation point at a distance $r_{grav.}$ from point P. The following mass equivalent produces the desired results:

$$M_{grav.} = \frac{F \times r^2}{\gamma \times m} = \frac{F \times r_{grav.}^2}{\gamma \times m} \qquad \text{r turns into } r_{grav.} \qquad \textbf{(F8)}$$

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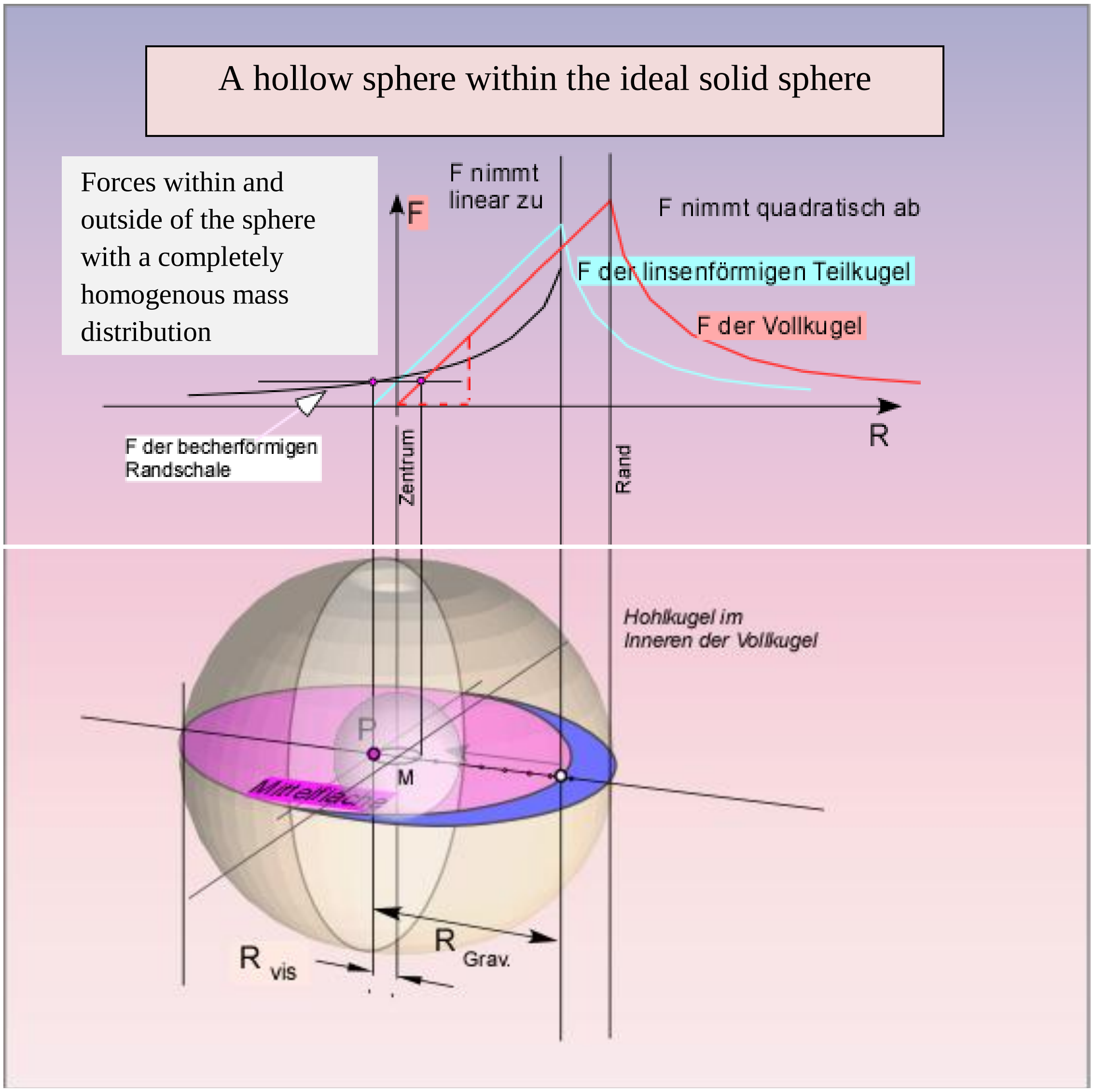
Figure 11



The discrete calculation cannot be done with a single formula. A body has to be divided in the largest possible number of single points. Such a grid lining produces a number of single calculations that need to be added for a result. Thanks to computers, this is easily done with an EXCEL data sheet. For the examples in this paper, the EXCEL program calculates with 5065 mass points and 10 measuring points (+1 center point). The program has to determine the forces of 5064 mass points on each of the measuring points, which leads to a sum of over 50,000 single calculations. Because of the gridding, the results are off by about 1%, which is usually a tolerable discrepancy.

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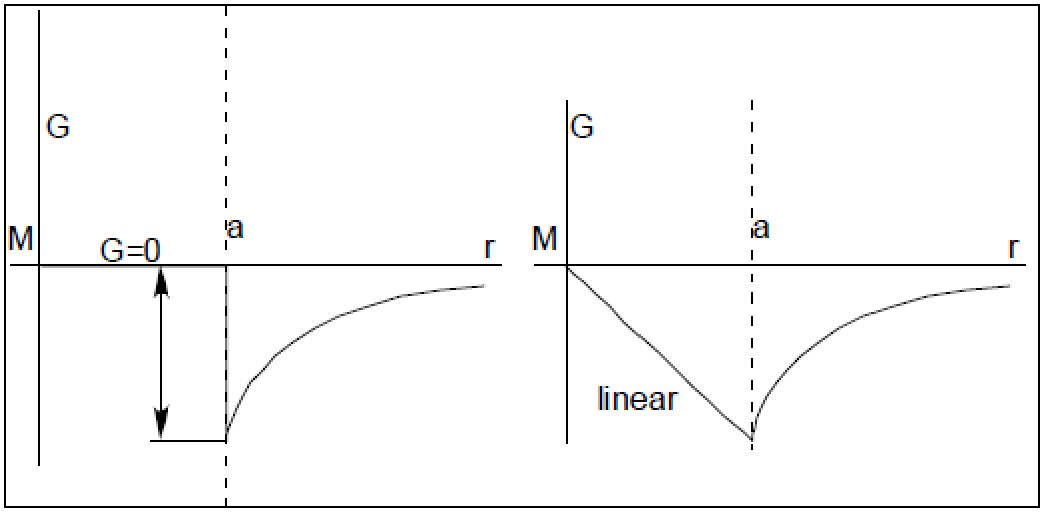
Figure 12



The two graphs in *Figure 12* represent the results of a hollow sphere (on the left) and a solid sphere calculation (on the right). The gravity on P on the inside as well as on the outside are illustrated. These results are similar to those determined with the integral center-oriented calculation method.

The discrete measuring point-oriented and integral center-oriented calculation methods produce the same results for gravitational forces inside and outside a solid homogenous or hollow sphere.

The labor, however, for the discrete calculation is more intense. Nevertheless, the discrete method leaves the option to change the masses within their symmetry. This produces mistakes in the integral calculation (…*errors occur if the examined body has a different symmetry* (p.7)). A comparison of both methods raises the question why the results are equal, even though the start variables (r, M) could not be more different.

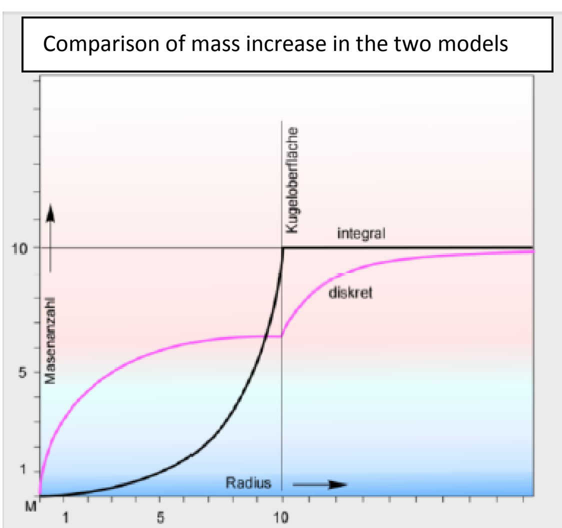


Figure 13

The two curves for the gravity exerting masses on P are very different in and around the solid sphere. The discretely determined mass values approach the absolute mass value, but never reach it - even on outside of the sphere, because the forces of masses at 180° counteract each other.

Their counteraction changes depending on the angle. The integral curve reaches its maximum on the outside, where it is equal to the absolute mass of the sphere. Logically, with the different masses but equal results, the radius curves are different as well (*Figure 14*).

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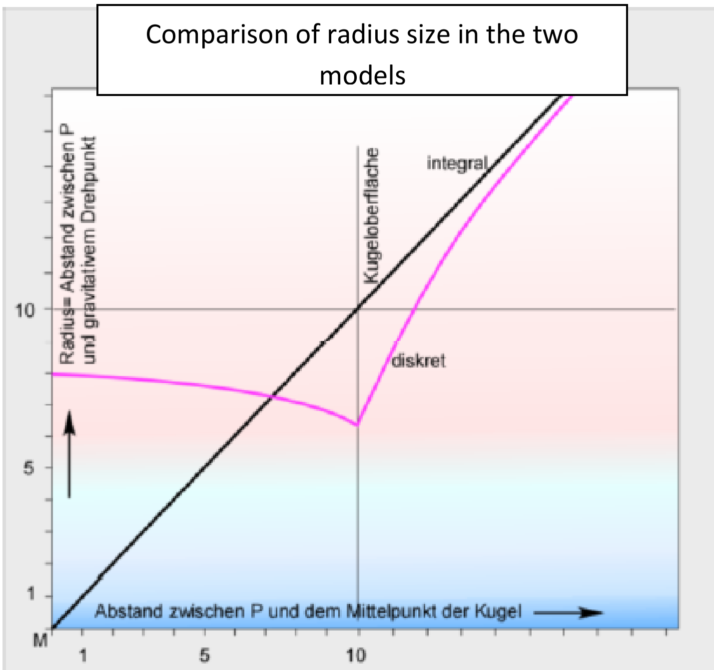
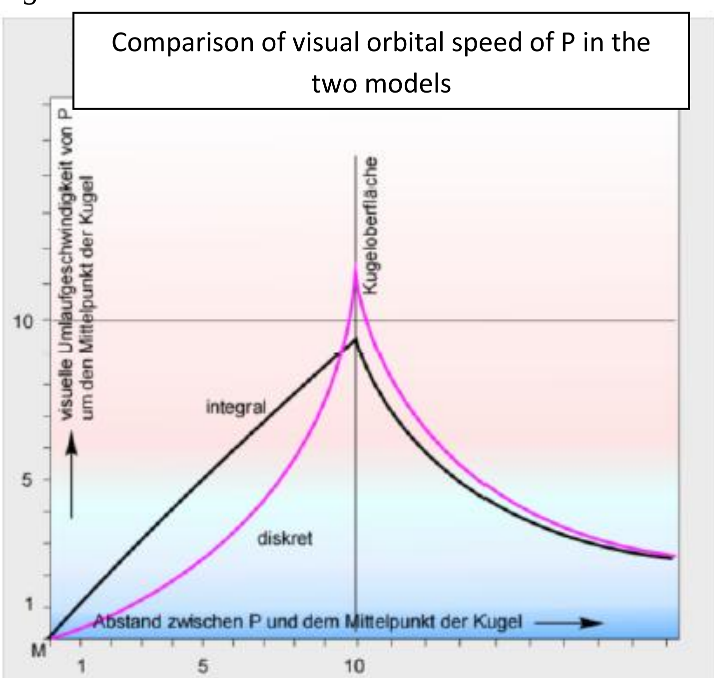
Figure 14



The radius of the integral calculation increases steadily and indicates the distance between point P and the center of the sphere. The radius of the discrete calculation shows the distance between point P and the gravitational rotation point of all relevant masses. This radius r_{grav} is different from the visual radius g_{vis}, because P moves on a libration track, not the gravitational orbit. This libration orbit is only the visible orbit of P around the center of the sphere.

Figure 15



Consequently, any further calculation of these different values for mass and radius produces different results, for example, the determination of the visual orbital speed of P around the center of the sphere.
The speed of P calculated with the discrete method is greater at the edge of the sphere than that of the integral method.
This does not change even if more mass is added to the sphere's center and the masses are not distributed evenly anymore.

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Summary

A comparison of the two calculation methods reveals that they produce different results, with the exception of the gravitational force in a homogenous symmetrical sphere. Even though the integral calculation method is the easier and faster method, it has its limitation: The values for the gravitational force are only correct when the body is absolutely homogenous and symmetrical.

The discrete calculation method delivers correct results for any sphere symmetrical body, regardless of its mass distribution. In addition, the basis values for the gravity calculation (mass and radius) are different in either model, which means that any further calculation will most likely lead to different results.

The discrete method has the advantage of using the gravitational orbit for its calculation of the visual orbital speed. Masses on either orbit take the same time for one complete rotation, but their speed is different, which has far reaching consequences. If the mass of a distant galaxy is determined over its visual orbital speed, the integral method delivers a result that is too big. This error in calculation is the basis for the crazy assumption that an invisible dark matter exists (Krause, 2005b).

Why do both methods produce correct results for the gravity in a homogenous symmetrical sphere?

The integral formula is based on two major mistakes. The mathematical steps are correct, but two errors in logic cause a faulty derivation. One of these errors is very field specific: The masses are combined over their potential *V* and not over their gravitational force *G* or *F*. Such a calculation ignores the direction of force until the formula is transformed into a vector formula with a common direction e_r. (See **(F4)** in **(F5)** and **(F6)**). This transformation is incorrect, because the direction of force pertaining to point P cannot be derived from the formula for the potential.

The following thorough examination of the formulas points out the mistake:

- The first error is done in formula (F1.1), where the gravitational force of a strip (*F* or *G*) (the gravity potential *V*) on point P is assumed to be too large. At this point, the counteraction of masses is not included in the calculation. The reduction of forces (of the potential) should have been done with the help of a parallelogram of forces. As P moves closer to the outside surface of the sphere, the angle ($2 \times \alpha$) between the masses grows. Consequently, the counteraction between masses increases (*Figure 16*).
  By neglecting this counteraction, the potential of each strip is set too high. A correct calculation can be found in the “Many-body problem” by Krause (2005a) in this forum.

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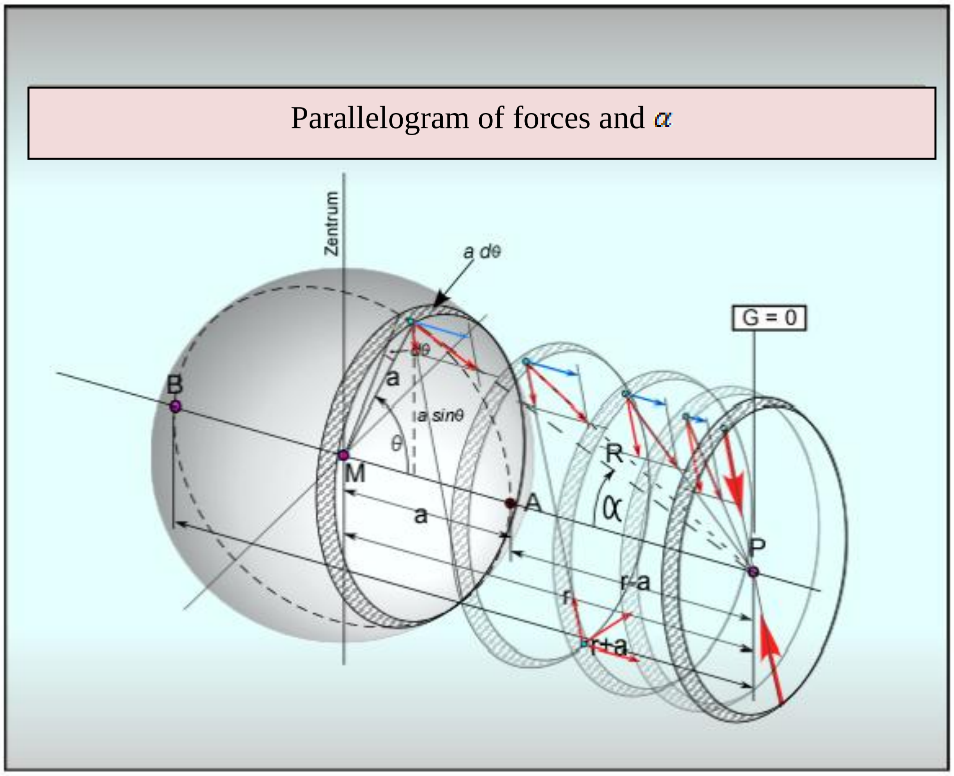
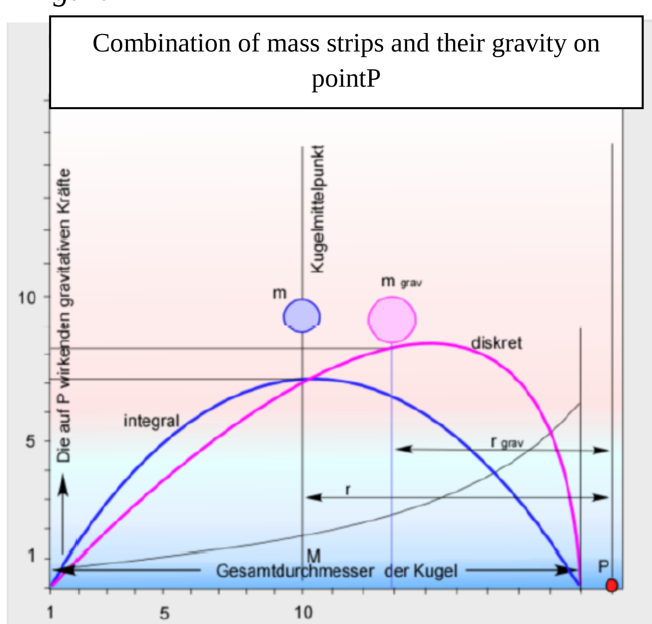
Figure 16



Only a part of the forces influences point P gravitationally (illustrated by the blue arrows). The same applies to the lower half of the sphere and to the strips that are on the sphere. If, for example, P lies in the center of the strip (to the right), all the forces counteract each other (*G*=0).

- The second mistake is located in formula **(F3)**, where the forces are too small. The reason for this lies in the summarizing of potentials (forces of all strips) in the center of sphere masses (blue circle in *Figure 17*), not in the gravitational center of forces (pink circle in *Figure 17*). This mistake is specific to the field of gravity calculations as well. By combining the masses in the center of the sphere, the distance from P increases more than it would if the masses were combined in the gravitational center. r versus r_{grav} .

Figure 17



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Consequently, the potential is too small. How to add the single masses correctly is explained in the “Many-body Problem” by Krause (2005a).

- Both mistakes incidentally cancel each other out throughout the whole calculation. The end result is correct, even though the calculation is faulty.

> Due to the facts that the integral center-oriented method delivers different results than the discrete method in all parameters, and that it relies on a faulty calculation for the determination of gravitational forces, it is useless in the field of cosmic calculations.

Derivation of gravity in a ‘hollow’ plane

The derivation of gravity in a plane is possible with the help of the sphere model. The first step is to take an ideal hollow sphere with a homogenous mass distribution on its surface. The gravitational forces in the center of the sphere are nonexistent. Point P divides the sphere into two halves if a line orthogonal to *r* with the distance *r* is drawn through P. All the masses in the segment facing away from the center of the sphere exert a gravitational force that pulls P away from the center. This gravity shall be called negative gravity. The masses in the opposite sphere segment pull P toward the center, which shall be called positive gravity. The positive and negative gravities are identical in strength, but pull P in opposite directions. The counteraction of gravities results in a nonexistent gravity for point P.

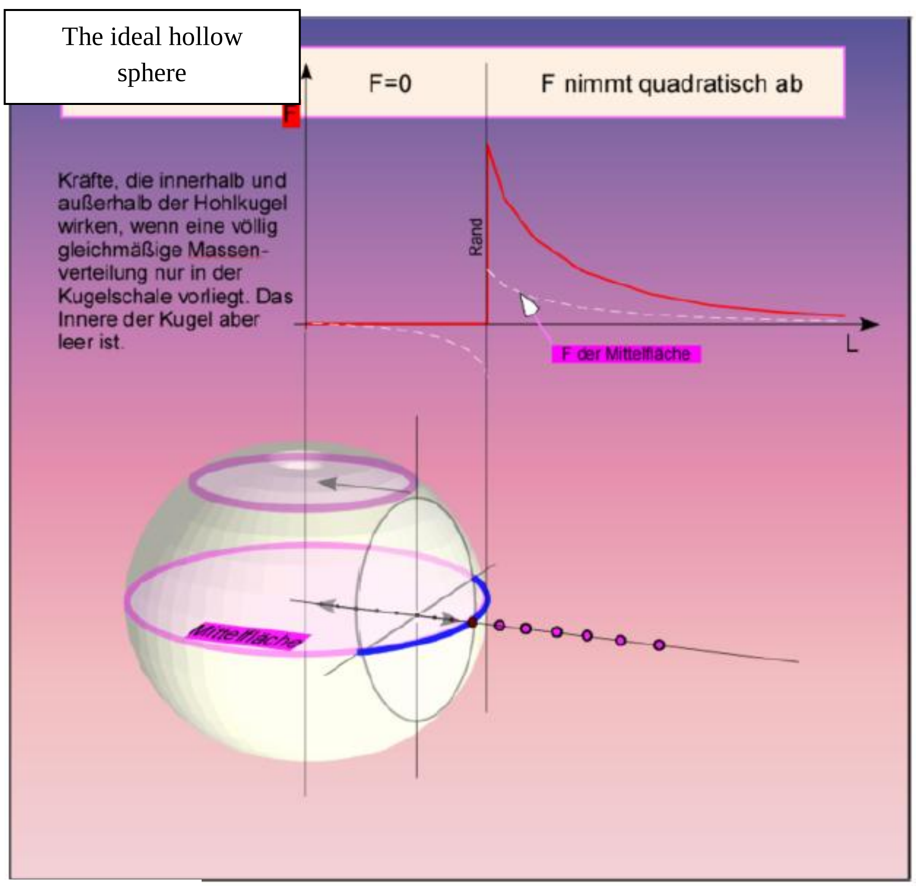

Figure 18

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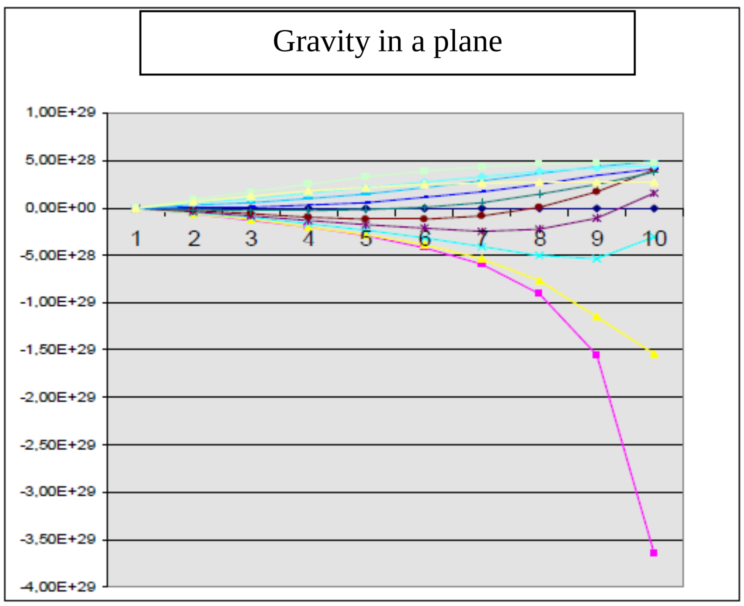
The second step is to divide the sphere into thin horizontal planes. Two of these planes, which are in actuality only mass rings, are used for the following evaluation. One of the plane is the mid plane. The center of the sphere is also the center of this plane. The other plane is close to the top of the sphere. Its radius is smaller than that of the radius emminating from point P. The gravity of all planes combined on P is zero, since the hollow sphere is the sum of all planes.
The gravitational forces pertaining to P within the small plane are directed toward the center, which is illustrated by the arrow pointing in. Consequently, P is exposed to a positive gravitational pull.
The mid plane, however, pulls point P toward the center (pink) and toward the edge (blue). Which of these two forces in the mid plane is stronger? Since the sum of all forces on P is zero, the force of the mid plane on P must be negative (toward the edge). The reason for this is founded in the small plane. For the forces to add up to zero, and the small plane only exerting positive gravity, the mid plane's gravity has to be negative.

Figure 19



Therefore, the gravity of different planes is unequal. *Figure 19* compares the forces of ten planes above and below the mid plane. The sum of their forces on P is zero. The mid plane with point P is represented by the pink curve. Clearly, the majority of planes exert positive gravity, while the pink (and yellow) curve delivers the negative gravity to facilitate a combined force of zero. It should not be difficult to see that the calculation of gravity in a sphere cannot be transferred to a plane.

A plane with its entire mass at its edge has different gravitational characteristics than a homogenous hollow sphere. While the gravity on a point P within the hollow sphere is zero, the gravity within a plane is negative (a point P is pulled to the edge).

The negative gravity on P increases with a growing distance from the plane's center.

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Derivation of gravity in a 'solid' plane

The derivation of gravity in a solid plane is similar to that of a hollow plane. The solid sphere is divided into a number of thin planes. The mid plane with point P and a plane with a smaller radius than that of P are evaluated. Because of the homogenous mass distribution throughout the plane, it is easy to determine the mass needed to counteract the blue shaded area of negative gravity. The pink area separated by the blue dotted line together with the blue area shows the lens shape in which gravity on P is zero. Subtracting this lens from the rest of the mid plane leaves an area of solely positive gravity on P, with a gravitational center that is different from the center of the plane.

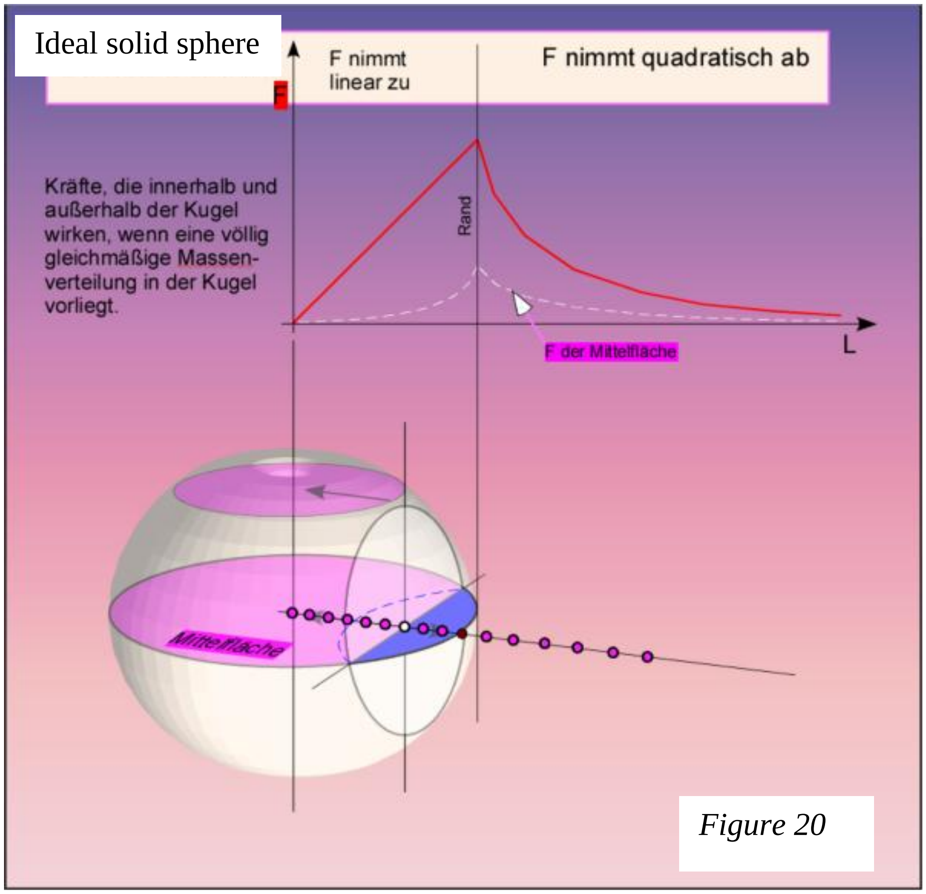

Figure 20

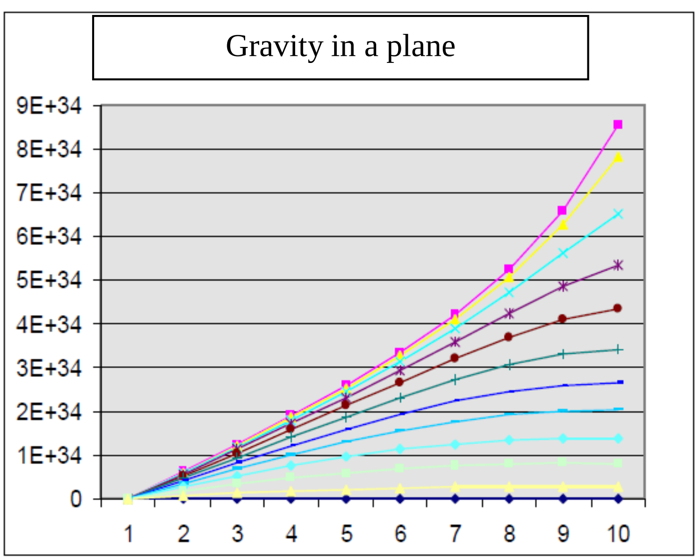

Figure 21 compares the forces of ten planes above and below the mid plane. The sum of their forces on P in the solid sphere is linearly increasing as seen in *Figure 12* on the right side. The pink curve represents the mid plane with point P.

The slope of gravity within a plane, however, is not linear like that of a solid sphere,

because the neutral area decreases non-linear as point P moves closer to the

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edge. If the gravity on the outside of the solid mid plane is calculated discretely, it also decreases, but not equal to that of a solid sphere. The gravitational center (of the plane) lies closer to point P (discretely calculated) than the center of the plane. Consequently, the gravitational force on point P in a plane is much stronger than that in a sphere.

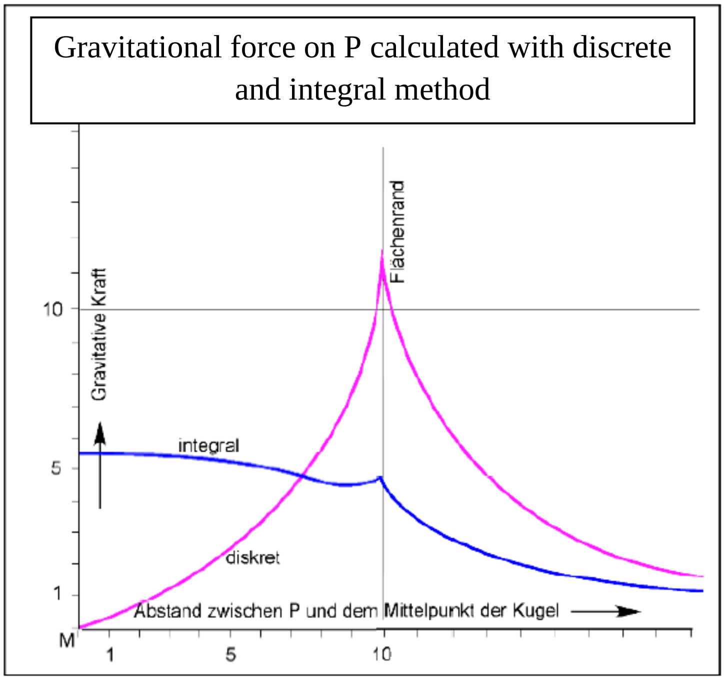
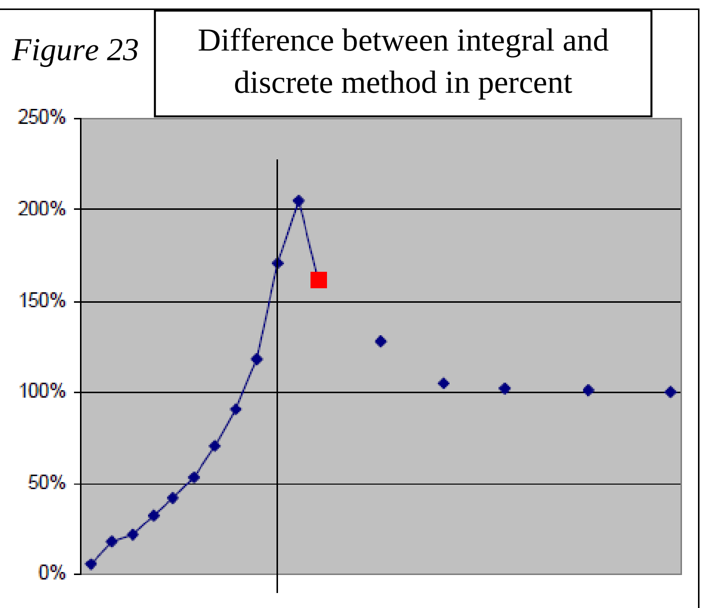


Figure 22
The gravitational forces n a point P within and outside of a solid plane as calculated with integral and discrete methods have absolutely no similarities. The pink curve illustrates the gravity on a point P determined with discrete calculation, while the blue curve represents the results of an integral calculation.

Comparison of gravity in the integral and discrete model

While at least the results of the gravity calculation in and around a homogenous symmetrical sphere were equal, a comparison of integral and discrete method applied to a plane reveals that none of the values for gravity *F*, mass *m*, and radius *r* are similar. The enormous differences between the two methods are illustrated in *Figure 23*.

Figure 23



The 100% line represents the values of gravity determined with the integral method. The values above this line show the gravitational forces for a point P at the edge of the plane determined with the discrete method, and the values below the 100% line stand for the forces for a point within the plane also determined with the discrete method. The single points in

Figure 23 represent different distances from the center of the plane, beginning on the left. At point 11, the highest marker, all masses in the discrete model exert gravitational force on P. (the grid lining prevents this in point 10). The red marker is the first point on the outside of the plane at a distance of 12, followed by 15, 30, 45, 60, and 100.

The discretely determined values for the gravitational force close to the center of a homogenous field are considerably smaller than those calculated with the integral method. Once the point P is moved close to the edge of the plane, the values of the discrete method increase much faster and even double than those of the integral calculation. On the outside of the plane, the values of the discrete calculation approach, but never reach the lower values of the integral method.

The integral values for the gravitational forces of a hollow and solid sphere are wrong.

Further application of the discrete gravity calculation

Three examples of the errors that occur with integral calculation are the distance calculation of pioneer probes, the light deviation of galactic lenses, and the debatable dark matter.

- ✓ A faulty calculation is a very likely explanation for the unexpected, hesitant movement of the two pioneer probes out of our solar system. If all masses of our solar system (perhaps even the mass of the Oort cloud) are entered in a discrete calculation model, the gravitational factor increases by 0.0064% compared to the integral calculation method. This sufficiently explains the puzzling brake-effect on the pioneer probe. Hence, there is no need to postulate a new energy (NZ Online, 2002)!
- ✓ The tendency of distant galactic gravity lenses to deflect light more intensely than calculated is also easily explained. The gravitational force determined with integral calculation is too low! A discrete calculation of gravitational forces results in a mass equivalent twice as high as previously determined. In addition, the integral calculation method uses point masses that are too small (Wambsganss & Schmidt, 2005).
- ✓ The dark matter error is more prevalent in the calculation of field or surface than the calculation of spheres, because of the differences in gravitational forces. If the mass of a distant field galaxy is determined over the visual orbit speed of its masses (on libration tracks), the result of such an integral calculation is always too large. On one hand, the base mass is assumed to be too big, and on the other hand, the gravitational orbit is confused with the libration track. This double error leads unavoidably to the incorrect assumption of an invisible dark matter (Krause, 2005a).

Three observable – until now- inexplicable phenomenon can be explained with a discrete calculation of values and without the assumption of dark matter.

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